Equivariant Cohomology

6/2007

Ref. Atiyah-Bott, Topology (1984)

Review cohomology $M \rightsquigarrow \underset{\mathbb{Z}\text{-mod}}{H^*(M)}$

- homotopy eq.
- $f: N \hookrightarrow M$

  $f_*: H^*(N) \longrightarrow H^{*+q}(M)$ ← codim.

  $\cup$ (Thom class)
- $f^* f_* 1 = \mathrm{Euler}(\nu_{N/M})$ ← normal bdl.
- $f: N \xrightarrow{\text{fiber bundle}} \!\!\!\rightarrow M$

  $f_* = \int_{\text{fiber}}$

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§2 Equivariant Theory Reviewed

cpt conn. Lie gp. $G \curvearrowright M$

Want FREE action $\Rightarrow H^*(M/G)$

(i.e. mod out symmetry)

Non-free $\Rightarrow$ enlarge (homotopy) to free

$G \curvearrowright pt. \sim G \curvearrowright \underset{\substack{\wr \\ pt.}}{EG} \Rightarrow H^*(\underbrace{EG/G}_{BG.})$

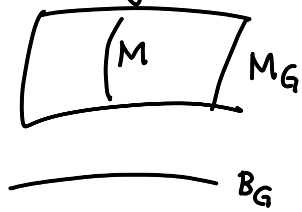
(Universal) pr. G-bdl

$$G \longrightarrow \underset{\displaystyle EG}{\overset{*}{\wr}} \!\! \longrightarrow BG$$

Def. $H^*_G(M) := H^*\big(\underbrace{\tfrac{M \times EG}{G}}_{M_G}\big)$

$$M \xrightarrow{i} M_G \longrightarrow B_G$$

fiber bundle



- $\rightsquigarrow \; H^*(B_G) \to H^*(M_G)$

  $\Rightarrow \; H^*_G(M)$: $H^*(B_G)$-mod.

- $H^*_G(M) \longrightarrow H^*(M)$

  (equivar. extn.)

- $H^*_G(pt) = H^*(B_G) = (S\mathfrak{g}^*)^{Ad(G)} = (S\,\mathfrak{t}^*)^W$

  $H^*_T(pt) = H^*(\prod^{\ell} \mathbb{C}P^\infty) = \mathbb{R}[u_1, \dots, u_\ell]$

- Localization

  $$T \curvearrowright M \xrightarrow{\pi} pt.$$

  $$\phi \in H^*_T(M) \xrightarrow{\int_{M_T/B_T}} H^*_T(pt) = \mathbb{R}[u_1, \dots, u_\ell]$$

  $$\Rightarrow \int_{M_T/B_T} \phi = \int_{(M^T)_T/B_T} \left(\frac{\phi|_{M^T}}{\mathrm{Euler}(\nu_{M^T/M})} \right)$$

- $$\begin{array}{ccc} \phi & H^*_T(M) \longrightarrow H^*_T(pt) & \\ \downarrow & \downarrow \qquad\qquad \downarrow \text{forget } u\text{'s} & \\ \phi_0 & H^*(M) \longrightarrow H^*(pt) = \mathbb{R} & \end{array}$$

  $\int_M \phi_0 \in \mathbb{R}$

  $\parallel$

  $\int_{M_T/B_T} \phi$ then set u's $= 0$.

§4 Equivar. deRham theory

G cpt conn. Lie gp.

Thm. $H^*(G) = (\Lambda \mathfrak{g}^*)^{ad(\mathfrak{g})}$

Remark: $G \curvearrowright (M, g) \Rightarrow$ harmonic forms are G-inv.

(key: $G \curvearrowright H^\cdot(M, \mathbb{Z})$ trivial)

Lemma $G \curvearrowright M \Rightarrow H^\cdot(\Omega^\cdot(M), d) = H^\cdot(\Omega^\cdot(M)^G, d)$

Pf: by averaging $\int_G \square \, dg$.

Pf: $H^*(G, \mathbb{R}) = H^*(\Omega^*(G), d) = H^*(\underbrace{\Omega^*(G)^{G_L}}_{\Lambda^\cdot \mathfrak{g}^*}, d)$ Lie alg. cohomology

left action

Here

$$d : \Lambda^k \mathfrak{g}^* \longrightarrow \Lambda^{k+1} \mathfrak{g}^*$$

determined by $k=1$ case, which is adjoint to

$$[\ \ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \quad \text{Lie bracket.}$$

$\left(\because \quad G \leq \mathrm{Diff}(G) \overset{[\ ]}{\curvearrowright} \Gamma(G, T_G) \overset{\text{dual}}{\rightsquigarrow} d : \Omega^k(G) \to \Omega^{k+1}(G) \right)$

$$H^\cdot(G, \mathbb{R}) = H^\cdot(\Omega^\cdot(G)^{G_L \times G_R}, d)$$
$$= H^\cdot((\Lambda^\cdot \mathfrak{g}^*)^{Ad\,\mathfrak{g}}, d) \qquad \left(\begin{smallmatrix}\text{Left} + \\ \text{right inv.}\end{smallmatrix} \equiv \begin{smallmatrix}\text{Left} + \\ \text{adj. inv.}\end{smallmatrix} \right)$$
$$= (\Lambda^\cdot \mathfrak{g}^*)^{Ad\,\mathfrak{g}}$$

Claim: $d = 0$ on $(\Lambda^\cdot \mathfrak{g})^{Ad\,\mathfrak{g}}$

Consider: $\iota : G \longrightarrow G$
$g \mapsto g^{-1}$

$$\Rightarrow \begin{array}{ccc} \iota^* : \Omega^*(G) & \longrightarrow & \Omega^*(G) \\ \cup & & \cup \\ \Omega^*(G)^{G_L} & \longrightarrow & \Omega^\cdot(G)^{G_R} \end{array}$$

In particular,

$$\iota^* : \underbrace{\Omega^*(G)^{G_L \times G_R}}_{(\Lambda^\cdot \mathfrak{g}^*)^{\mathrm{ad}\,\mathfrak{g}}} \longrightarrow \underbrace{\Omega^*(G)^{G_R \times G_L}}_{(\Lambda^\cdot \mathfrak{g}^*)^{\mathrm{ad}\,\mathfrak{g}}}$$

Note: $\iota_* = (-1) : T_e G \longrightarrow T_e G$

$\Rightarrow \quad \iota^* = (-1)^k$ on $(\Lambda^k \mathfrak{g}^*)^{\mathrm{ad}\,\mathfrak{g}}$

$\Rightarrow \quad d = 0 \qquad (\because [d, \iota^*] = 0)$.

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$$G \longrightarrow EG \longrightarrow BG$$

Qu: $H^\cdot(BG) = ?$

Need: Describe BG, or $G \overset{\text{free}}{\curvearrowright} EG \sim *$

Can use ANY such model to describe BG (e.g. cosimplicial construction).

Also, can use ANY suitable complex to compute $H^\cdot(BG)$.

$$\underset{\text{(Hopf alg.)}}{(\Lambda^\cdot \mathfrak{g}^*, d)} \overset{\text{"free act}^{\text{n}}\text{"}}{\subset} (W, D) \overset{\text{quasi-isom}}{\sim} \mathbb{R}$$

(i) Weyl model, (ii) Cartan model, (iii) BRST model.

- $H^*(EG) = \mathbb{R}$ ($\because EG \sim pt$).

Weyl alg. $W(\mathfrak{g}) = \underbrace{\Lambda \mathfrak{g}^*}_{\substack{\theta^\alpha \\ \deg 1}} \otimes \underbrace{S \mathfrak{g}^*}_{\substack{u^\alpha \\ \deg 2}} = \mathbb{R}[\theta^\alpha, u^\alpha]$

Cartan-Maurer $d\theta^\alpha + \frac{1}{2} \underbrace{c^\alpha_{\beta\gamma}}_{\text{str. const.}} \theta^\beta \theta^\gamma = 0$

$D : W(\mathfrak{g}) \longrightarrow W(\mathfrak{g})$, $D^2 = 0$ (Jacobi id. for [ ])

$D\theta^\alpha + \frac{1}{2} c^\alpha_{\beta\gamma} \theta^\beta \theta^\gamma + u^\alpha = 0$

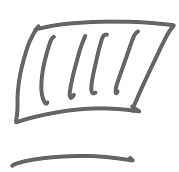
$D u^\alpha - c^\alpha_{\beta\gamma} u^\beta \theta^\gamma$

Note: $\underset{\substack{\wr \\ H^\cdot(G)}}{(\Lambda^\cdot \mathfrak{g}^*, d)} = (W(\mathfrak{g}), D)\big|_{u^\cdot = 0}$

Fact:

$$H^*(W(\mathfrak{g}), D) = \mathbb{R}.$$

$\Omega^*(G)$	Weyl model	$\Lambda^\cdot \mathfrak{g}^*$
$\Omega^*(EG)$		$W(\mathfrak{g})$
$\Omega^*(BG)$		?

Recall : $P \xrightarrow{\pi} M$  fiber bundle.

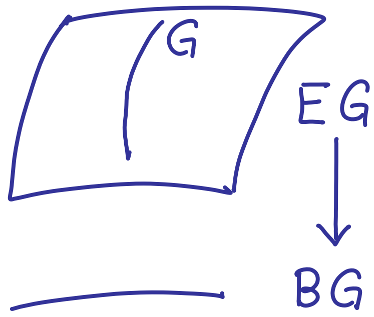
Basic forms.

$$\Omega^*(M) \xhookrightarrow{\pi^*} \Omega^*(P)$$

Image = ? $\Omega^*(P)_{basic} \ni \varphi$

(i.e. $\pi_* X = 0$)

$\iota_X \varphi = 0 = \mathcal{L}_X \varphi$ $\forall$ vertical v.f. X



$\Omega^*(EG) \sim W(\mathfrak{g})$

Ex: $W(\mathfrak{g})_{\text{basic}} = \left(S(\mathfrak{g}^*)\right)^{Ad(G)}$ $\qquad D = 0$

(reason: On $\mathfrak{g}$, dual base e_α $(\iota_{e_\alpha}\theta^\beta = \delta_\alpha^\beta)$
$\iota_{e_\alpha} u^\beta = 0$, $\mathcal{L}_{e_\alpha} = \iota_{e_\alpha} D + D\,\iota_{e_\alpha}$)

$$\Rightarrow \quad H^*(BG) = (S\mathfrak{g}^*)^{AdG}$$

$$H^*(BG) = H^*_G(pt)$$

$G \curvearrowright M$

$$\rightsquigarrow \quad G \longrightarrow M \times EG \longrightarrow M_G \quad \text{fiber bdl.}$$

$$\Omega^*(M) \otimes W(\mathfrak{g}) \qquad \underbrace{(\Omega^*(M) \otimes W(\mathfrak{g}))_{\text{basic}}}_{\Omega^*_G(M)}$$

Theorem: $H^*_G(M) = H^*(\Omega^*_G(M), D)$

Eg. $S^1 \curvearrowright M$ $\quad H^*_{S^1}(M) = ?$

$$\Omega^*(M) \otimes \underbrace{W(\mathfrak{g})}_{\mathbb{R}[\theta, u]} \qquad \theta^2 = 0$$

$$\varphi = \sum a_k u^k + \sum b_\ell u^\ell \theta \qquad a_k, b_\ell \in \Omega^*(M)$$

φ : basic

$$\overset{\Delta}{\Longleftrightarrow} \quad \iota_X \varphi = 0 = \mathcal{L}_X \varphi \qquad X \in \operatorname{Lie} S^1 = \mathbb{R} \quad (\iota_X \theta = 1)$$

$$\overset{\text{Ex.}}{\Longleftrightarrow} \quad \mathcal{L}_X a_k = 0, \quad b_k = -\iota_X a_k$$

Let $\Omega^*(M)^{S^1} := \Omega^*(M) \cap \operatorname{Ker}(\mathcal{L}_X)$

i.e. invariant forms.

Ex: $\lambda : \Omega^*(M)^{S^1}[u] \longrightarrow \Omega^*(M) \otimes W(\mathfrak{g})$

$$a \mapsto a - (\iota_X a)\theta$$

$$u \mapsto u$$

induces $\Omega^*(M)^{S^1}[u] \underset{\lambda}{\overset{\simeq}{\rightleftarrows}} \Omega^*_{S^1}(M)$

$$\underset{d + \iota_X u}{\overset{\shortparallel}{d_X}} \longleftrightarrow D$$

§6 Relations w/ Moment Map

$G \curvearrowright (M, \omega)$ Symplectic.

Prop

$\exists\ \mu : M \to \mathfrak{g}^*$ $\iff$ $[\omega - \mu] \in H_G^*(M)$

moment map equiv. ext^n of ω

Eg. $G = S^1$ $\quad x \in \mathrm{Lie}\, S^1 = \mathbb{R}$

$\mu : M \longrightarrow \mathbb{R}$

$$d_x(\omega - \mu u) = (d + \iota_x u)(\omega - \mu u)$$
$$= (d\omega) + \underline{(\iota_x\omega - d\mu)}u - (\iota_x\mu)u^2$$
$$= 0$$

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§7 Relation w/ DH formula.

$$T \curvearrowright (M, \omega) \xrightarrow{\mu} \mathfrak{t}^* = \mathbb{R}^\ell$$

- (Convexity) $\quad \mu(M) \subseteq \mathbb{R}^\ell$

  is convex polyhedron.

  ($\Rightarrow$ $\mu_*(\omega^n/n!)$ supp is convex)

- DH. $\quad \mu_*(\omega^n/n!)$ piecewise poly. measure

$$S^1 \curvearrowright (M,\omega) \xrightarrow{f} \mathbb{R}$$

$$\iota_X \omega = df$$

$$H^*_{S^1}(M) \longrightarrow H^*(M)$$

$$[\omega - f u] \mapsto [\omega]$$

$$e^{\omega - fu}$$

Localization: $M \xrightarrow{\pi} pt.$

$$\pi_*(e^{\omega - fu}) = \int_{M^{S^1}} \frac{(e^{\omega - fu})|_{M^{S^1}}}{\mathrm{Euler}(\nu_{M^{S^1}/M})}$$

$$\parallel$$

$$\int_M e^{-fu} \frac{\omega^n}{n!} \in H^*_{S^1}(pt) = \mathbb{R}[u]$$

Say $M^{S^1} = \{P\}$ $(\Rightarrow e^{\omega}|_{M^{S^1}} = 1,\ e^{-fu}|_{M^{S^1}} = e^{-f(p)u})$

$\mathrm{Euler}(\nu_P) = ?$

$$S^1 \curvearrowright T_P M = \bigoplus_{k=1}^n V_k \qquad \text{rotation.}$$

$$m_k \in \mathbb{Z}$$

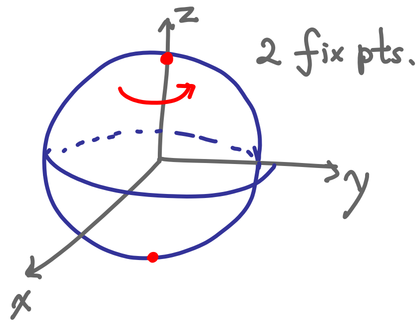
$$\Rightarrow \mathrm{Euler}(\nu_P) = \pm \prod_{k=1}^n (m_k u)$$

i.e. $$\int_M e^{itf} \frac{\omega^n}{n!} = \sum_{P \in M^{S^1}} \left(\frac{i}{t}\right)^n \frac{e^{itf(P)}}{\pm \prod_{k=1}^n m_k(P)}$$

i.e. stationary phase approx is exact.

Eg. $S^1 \curvearrowright S^2 \subseteq \mathbb{R}^3$

$$\int_{S^2} e^{itz} \, dA$$



$$= \left(\frac{i}{t}\right)\left[e^{it} + e^{-it}\right]$$

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§ Localization.

For simplicity $G = S^1 \curvearrowright M$
$X \in \Gamma(M, TM)$

Case (1). free action

i.e. $X(x) \neq 0 \quad \forall x \in M$

$$\Leftrightarrow \quad \exists\, \theta \in \Omega^1(M)^{S^1}$$

$$\iota_X(\theta) = 1 \qquad (\mathcal{L}_X \theta = 0)$$

Lemma: $d_X(\eta) = 1 \Rightarrow H^\bullet_{S^1}(M) = 0$

$$\Rightarrow \int_{M_{S^1}/B_{S^1}} \varphi = 0 \qquad \forall\, d_X \varphi = 0$$

Pf of lemma: $\quad d_x\varphi = 0$

$$\Rightarrow \quad d_x(\varphi(u_0)) = (\cancel{d_x\varphi})(u_0) \pm \varphi\,\underbrace{(d_x(u_0))}_{1} = \varphi$$

i.e. all closed forms are exact.

$$\int (d_x\eta) = \int d\eta + u \int \iota_x \eta = 0.$$

Claim $\quad d_x\left(\dfrac{\theta}{d\theta + u}\right) = 1 \qquad$ (Exercise)

Here $\quad \dfrac{\theta}{d\theta+u} = \dfrac{\theta}{u}\left(1 - \dfrac{d\theta}{u} + \dfrac{(d\theta)^2}{u^2} - + \ldots\right)$

(need $u \neq 0$)

Note $\quad \dfrac{1}{d_x} = \dfrac{\theta}{d_x\theta} = \dfrac{\theta}{d\theta + u} \qquad (\because\ \iota_x\theta = 1).$

Case (2) $\qquad$ Not free.

$$M = \underbrace{\text{nbd}(M^{S^1})}_{\substack{\text{std model} \\ (\text{depending on} \\ \text{how } S^1 \curvearrowright T_pM)}} \cup (M \setminus \text{nbd})$$

$\rightsquigarrow$ localize the computation $\Rightarrow$ ✓ ▨